Proposed solutions for tutorial 6

Intermediate Microeconomics (UTS 23567)[*]

Preliminary and incomplete

Available at https://backwardinduction.blog/tutoring/

Office hours on Mondays from 9 am till 10 am in building 8 on level 9

Please whatsapp me on 0457871540 so I could meet you at the door, I don't have an internal phone

Also please whatsapp if you have questions, I won't be able to answer through whatsapp but I will give answer during office hours or, since you are very unlikely to have ask a unique question, in the beginning of next tutorial

Sergey V. Alexeev

1 of May, 2018

Navigation

Page numbers are clickable

[*]Questions for the tutorial were provided by Massimo Scotti, slide and textbook are by Nechyba (2016). Solutions and commentary are by Sergey Alexeev (e-mail: sergei.v.alexeev@gmail.com). (Btw this is a much better book Varian (1987))

Recall that by the definition TRS is something that preserves output

$$\Delta y \overset{set}{=} 0 \tag{S}$$

and we were taught to isolate an individual contributions of ℓ and k into y which allows to write this

$$\Delta y = MPL(\ell, k)\Delta\ell + MPK(\ell, k)\Delta k \overset{set}{=} 0$$

and drop Δy

$$MPL(\ell, k)\Delta\ell + MPK(\ell, k)\Delta k \overset{set}{=} 0$$

which upon rearranging is

$$\frac{\Delta k}{\Delta\ell} = -\frac{MPL(\ell, k)}{MPK(\ell, k)}$$

and we call those rations $TRS(\ell, k)$

$$\frac{\Delta k}{\Delta\ell} = -\frac{MPL(\ell, k)}{MPK(\ell, k)} \overset{def}{=} TRS(\ell, k)$$

For example if $TRS(\ell, k) = -23$ we can write

$$\frac{\Delta k}{\Delta\ell} = -23$$

which is identical to

$$\Delta k = -23\Delta\ell \tag{T}$$

recall that we got this ratio from (S), so it still preserves information from it, even if it is not obvious, which allows us to read the ratio as following.

A unit increase in l (i.e. $\Delta l = 1$) requires us to drop 23 units of k to have output unchanged

Note that it is easier to calculate $TRS(\ell, k)$ using marginal products, but easier to understand using the ratio of changes

$$\underbrace{\frac{\Delta k}{\Delta\ell}}_{\substack{\text{to}\\\text{understand}\\\text{TRS}}} = \underbrace{-\frac{MPL(\ell, k)}{MPK(\ell, k)}}_{\substack{\text{to calculate}\\\text{TRS}}} \overset{def}{=} TRS(\ell, k)$$

Also check out this thing to develop a better intuition for what is happening. I think UTS has licences for Mathematica for all students.

Question 1 (Perfect substitutes)

Suppose the technology of a producer is described by the following production function:

$$f(\ell, k) = 2\ell + k$$

1.a.

Write the equation of the isoquant with level of production 2

Answer to 1.a

$$f(\ell, k) = 2\ell + k = 2$$

Question 1 (Perfect substitutes)

Suppose the technology of a producer is described by the following production function:

$$f(\ell, k) = 2\ell + k$$

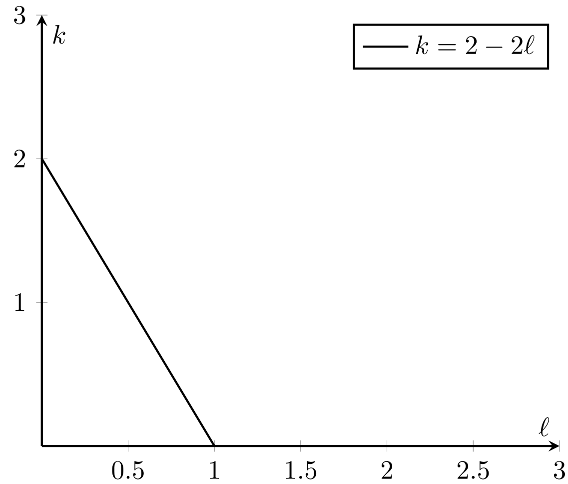
1.b

Draw the map of isoquants in a graph with ℓ on the horizontal axis and k on the vertical.

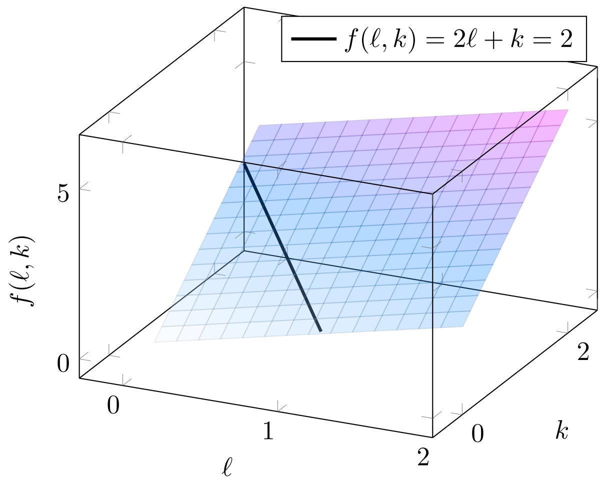
Answer to 1.b

$$f(\ell, k) = 2\ell + k = 2$$

$$k = 2 - 2\ell$$



As before keep in mind that we plot the above, but what we actually mean is this



Question 1 (Perfect substitutes)

Suppose the technology of a producer is described by the following production function:

$$f(\ell, k) = 2\ell + k$$

1.c

What can you say about the TRS in this case?

Answer to 1.c

$$\boxed{TRS(\ell, k) = -{}^{MPL}/_{MPK}}$$

$$TRS(\ell, k) = -{}^{MPL}/_{MPK} = -{}^{2}/_{1} = -2$$

Question 1 (Perfect substitutes)

Suppose the technology of a producer is described by the following production function:

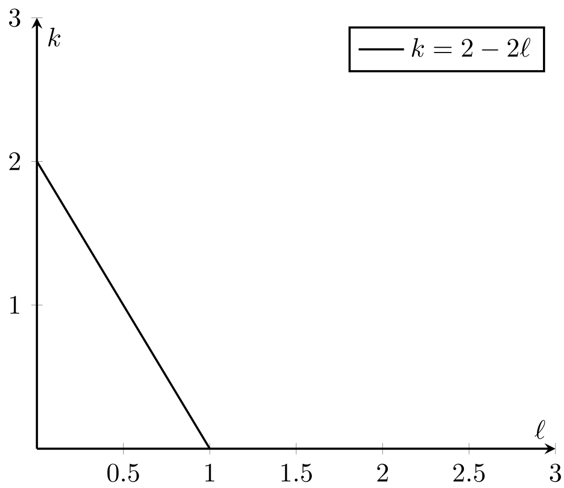
$$f(\ell, k) = 2\ell + k$$

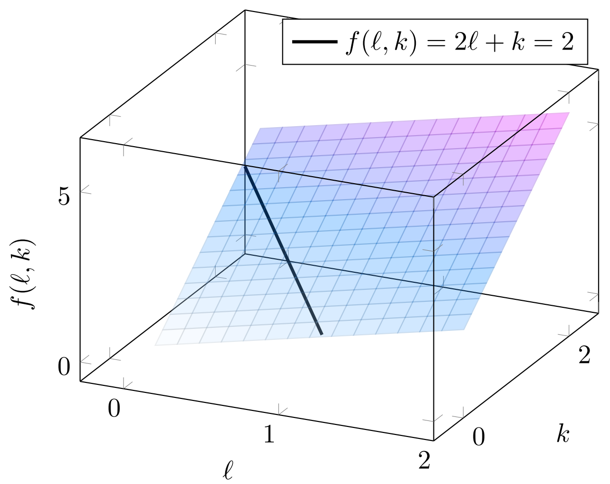
1.d

Derive the expression of the TRS and check that it is consistent with your graphical analysis.

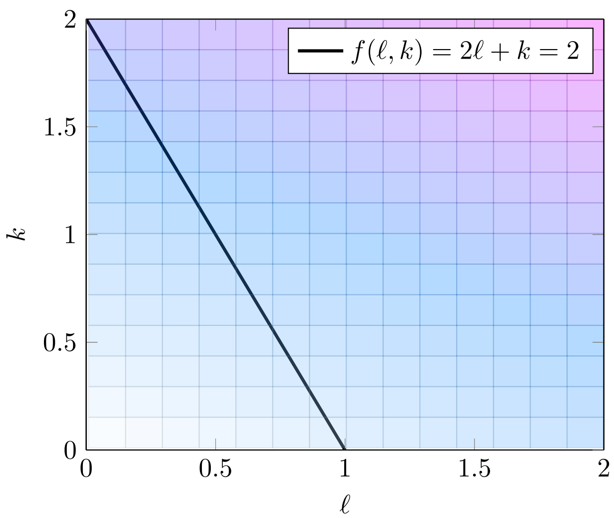
Answer to 1.d

$$TRS(\ell, k) = -{}^{MPL}/_{MPK} = -{}^{2}/_{1} = -2$$





To get the above picture we intersect $f(\ell, k)$ with level 2 in initial 3 dimensional space...



...and then express k as a function of ℓ which is identical to make a 90° rotation around the ℓ-axis which gives up with a view of the plot from top. Note that the legend tells us that the further we get to North East the higher the function value. And to get a figure above we just disregard the surface.

Question 1 (Perfect substitutes): Checklist

Suppose the technology of a producer is described by the following production function:

$$f(\ell, k) = 2\ell + k$$

1.a.
Write the equation of the isoquant with level of production 2.

1.b
Draw the map of isoquants in a graph with l on the horizontal axis and k on the vertical.

1.c
What can you say about the TRS in this case?

1.d
Derive the expression of the TRS and check that it is consistent with your graphical analysis

Question 2 (Perfect complements)

Now suppose the technology of a producer is described by the following production function:

$$f(\ell, k) = \min\{2\ell, k\}$$

2.a.

Write the equation of the isoquant with level of production 2.

Answer to 2.a

$$f(\ell, k) = \min\{2\ell, k\} = 2$$

to find ℓ and k we need to set

$$2\ell = 2$$

and

$$k = 2$$

which gives proportions that needs to be preserved to be at production level 2

$$\ell = 1, k = 2$$

There was nothing special about 2, to get on level 4 we need

$$\ell = 2, k = 4$$

and on 6

$$\ell = 3, k = 6$$

and so on

it also could be expressed as a simple ration

$$\frac{k}{\ell} = \frac{1}{2} \tag{P}$$

which says that inputs must go in this particular proportion.

Question 2 (Perfect complements)

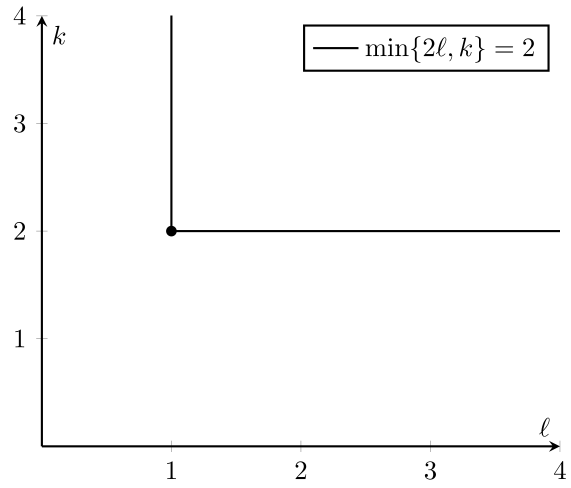
Now suppose the technology of a producer is described by the following production function:

$$f(\ell, k) = \min\{2\ell, k\}$$

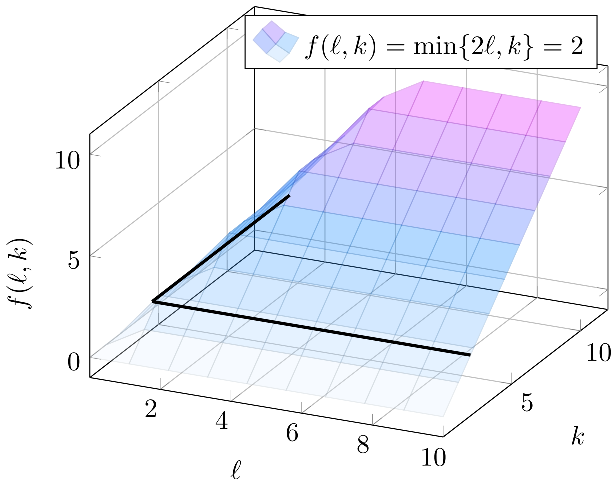
2.b

Draw the map of isoquants in a graph with ℓ on the horizontal axis and k on the vertical.

Answer to 2.b



Again note that we actually mean this



Question 2: PERFECT COMPLEMENTS

Now suppose the technology of a producer is described by the following production function:

$$f(\ell, k) = \min\{2\ell, k\}$$

2.c
What can you say about the TRS in this case?

Answer to 2.c

In the case of production function with $\min\{x_1, x_2\}$ a trade such as in (T) is impossible. Inputs should go in a very particular proportion (P)

Question 2 (Perfect complements): Checklist

Suppose the technology of a producer is described by the following production function:

$$f(\ell, k) = \min\{2\ell, k\}$$

2.a.

Write the equation of the isoquant with level of production 2.

2.b

Draw the map of isoquants in a graph with l on the horizontal axis and k on the vertical.

2.c

What can you say about the TRS in this case?

Question 3 (Cobb-Douglas)

- The two cases above represent extreme cases of technology.
- Typically, the degree of substitutability between labour and capital depends on the actual amount of labour and capital used.
- For example, consider the following Cobb-Douglas production function:

$$f(\ell, k) = \ell k$$

3.a.
Write the equation of the isoquant with level of production 2.

Answer to 3.a

$$f(\ell, k) = \ell k = 2$$

Question 3 (Cobb-Douglas)

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- For example, consider the following Cobb-Douglas production function:

$$f(\ell, k) = \ell k$$

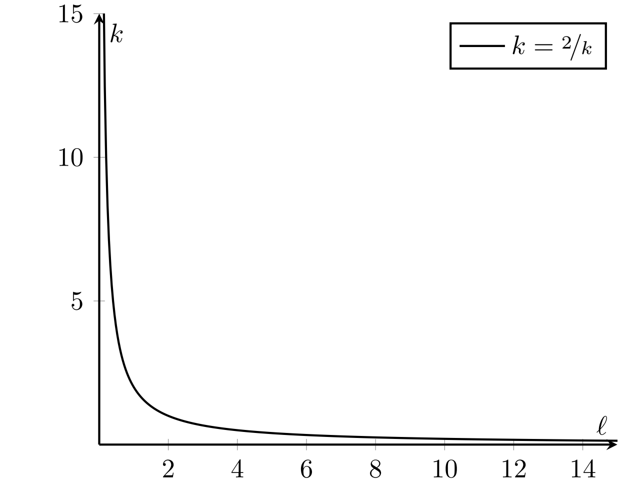
3.b

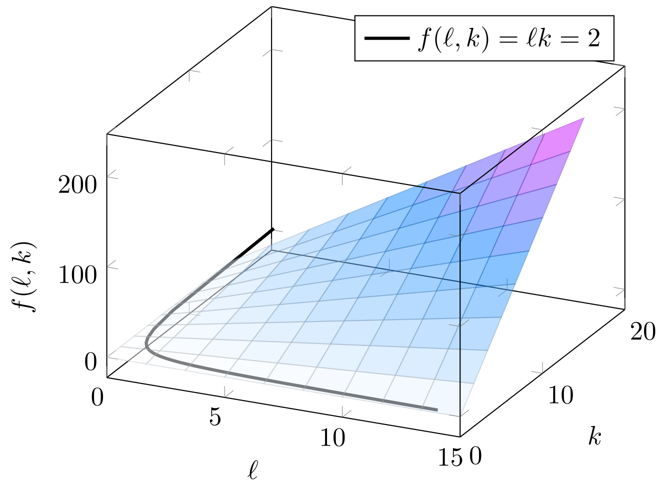
Draw the map of isoquants in a graph with ℓ on the horizontal axis and k on the vertical.

Answer to 3.b

$$f(\ell, k) = \ell k = 2$$

$$k = {}^{2}/_{\ell}$$





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$$f(\ell, k) = \ell k$$

3.c

What can you say about the TRS in this case?

Answer to 3.c

$$TRS(\ell, k) = -{}^{MPL}/_{MPK} = -{}^{k}/_{\ell}$$

Question 3 (Cobb-Douglas): Checklist

- The two cases above represent extreme cases of technology.
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Write the equation of the isoquant with level of production 2.

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Draw the map of isoquants in a graph with l on the horizontal axis and k on the vertical.

3.c

What can you say about the TRS in this case?

Cost minimization analytically

$$\min_{\ell,k} w\ell + rk$$
$$\text{s.t. } f(\ell,k) = x$$

$$\mathcal{L} = w\ell + rk - \lambda(f(\ell,k) - x)$$

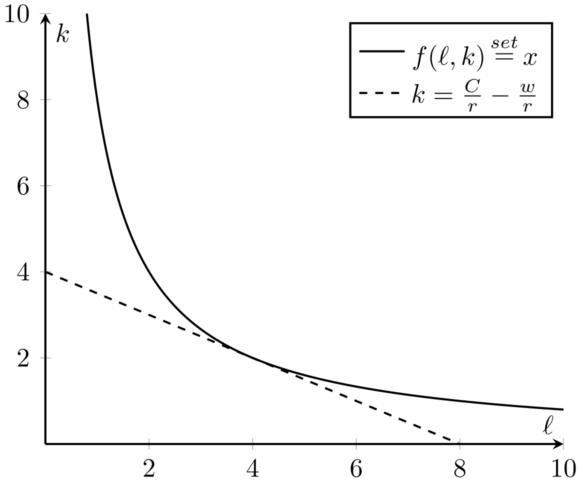
$$\mathcal{L}_\ell \Rightarrow w - \lambda\frac{\partial f(\ell,k)}{\partial \ell} = 0$$
$$\mathcal{L}_k \Rightarrow r - \lambda\frac{\partial f(\ell,k)}{\partial k} = 0$$
$$\mathcal{L}_\lambda \Rightarrow f(\ell,k) - x = 0$$

$$\frac{w}{r} = \frac{\partial f(\ell,k)/\partial \ell}{\partial f(\ell,k)/\partial k}$$

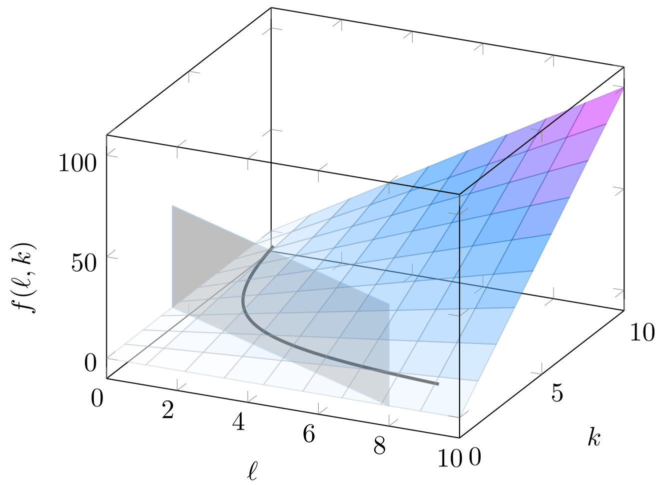
$$\boxed{TRS(\ell,k) = -w/r}$$

Cost minimization graphically

$$w\ell + rk \overset{set}{=} C$$
$$k = \frac{C}{r} - \frac{w}{r}\ell$$



(a) We fix a level of output at level y



then pick (ℓ, k) that manifests as shifting the dashed line (on left) or a plane (above) in a way that touches a desired level of production y and slopes of the associated isoquant and the isocost are equal. Thus again $\boxed{TRS(\ell,k) = -w/r}$

Question 4 (Cost minimization)

Suppose

$$w = 1$$

$$r = 2$$

For each of the following cases, find the least-cost combination of producing 8 units of output and provide a graphical representation

4.a. The producer is using technology

$$f(\ell, k) = \ell k$$

what is the least-cost combination of producing 8 units of output? Represent it graphically?

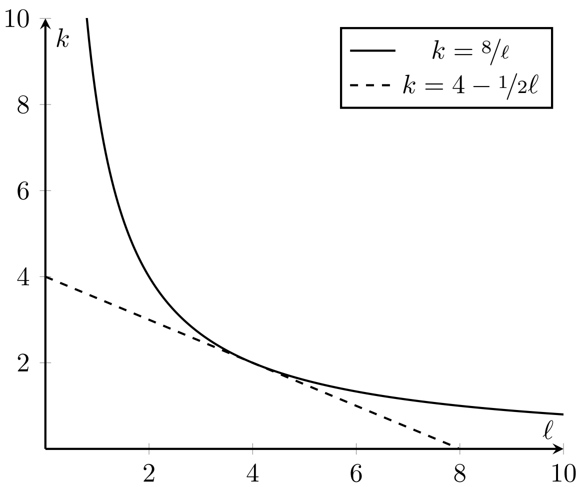
Answer to 4.a

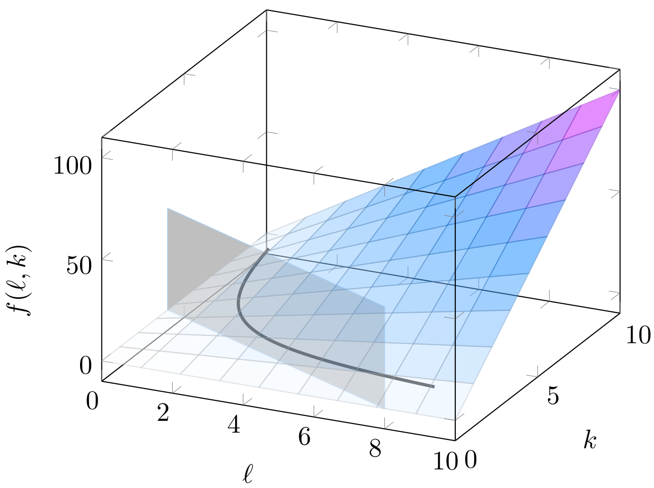
$$\boxed{\begin{cases} TRS(\ell,k) &= -{}^{w}/_{r} \\ f(\ell,k) &= x \end{cases}} \Rightarrow \begin{cases} -{}^{k}/_{\ell} &= -{}^{1}/_{2} \\ \ell k &= 8 \end{cases}$$

$$\Rightarrow \begin{cases} \ell &= 2k \\ \ell k &= 8 \end{cases}$$

$$\Rightarrow \begin{cases} \ell &= 2k \\ 2kk &= 8 \end{cases}$$

$$\Rightarrow \begin{cases} \ell &= 4 \\ k &= 2 \end{cases}$$





Question 4 (Cost minimization)

Suppose

$$w = 1$$

$$r = 2$$

For each of the following cases, find the least-cost combination of producing 8 units of output and provide a graphical representation

4.b.

$$f(\ell, k) = 2\ell + k$$

what is the least-cost combination of producing 8 units of output? Representit graphically?

Answer to 4.b

Obvious

Question 4 (Cost minimization)

Suppose

$$w = 1$$

$$r = 2$$

For each of the following cases, find the least-cost combination of producing 8 units of output and provide a graphical representation

4.c.

$$f(\ell, k) = \min\{2\ell, k\}$$

what is the least-cost combination of producing 8 units of output? Represent it graphically?

Answer to 4.c

To make

$$\min\{2\ell, k\} = 8$$

we need to set

$$2\ell = 8$$

$$k = 8$$

Question 4 (Cost minimization): checklist

Suppose

$$w = 1$$

$$r = 2$$

For each of the following cases, find the least-cost combination of producing 8 units of output and provide a graphical representation

4.a.

For each of the following cases, find the least-cost combination of producing 8 units of output and provide a graphical representation

$$f(\ell, k) = \ell k$$

what is the least-cost combination of producing 8 units of output? Represent it graphically?

4.b.

For each of the following cases, find the least-cost combination of producing 8 units of output and provide a graphical representation

$$f(\ell, k) = 2\ell + k$$

what is the least-cost combination of producing 8 units of output? Represent it graphically?

4.c.

For each of the following cases, find the least-cost combination of producing 8 units of output and provide a graphical representation

$$f(\ell, k) = \min\{2\ell, k\}$$

what is the least-cost combination of producing 8 units of output? Represent it graphically?

Question 5 (Understanding TRS and least-cost input combination)

- Suppose a producer's technology is described by a Cobb-Douglas production function.
- Further, suppose that

  the opportunity cost of labor equal to 2

  and

  the producer is producing a desired quantity x

  by using a bundle of inputs at which

  $$TRS = -3$$

- We know that in this case the producer would fail to minimize his production costs.
- Explain how the producer can save money by altering the composition of his current bundle of inputs.
- For the sake's of your reasoning, denote the current bundle with A and assume that

  the price of labor is 1

Answer to 5

We are asked to prove

$$\boxed{TRS(\ell, k) = -{}^{w}\!/_{r}}$$

by contradiction.

Idea is to show that all decisions with

$$TRS(\ell, k) \neq -{}^{w}\!/_{r}$$

are not optimal.

Recall that by the definition TRS is

$$\Delta y = MPL(\ell, k)\Delta\ell + MPK(\ell, k)\Delta k \overset{set}{=} 0$$

which upon rearranging is

$$\frac{\Delta k}{\Delta \ell} = -\frac{MPL(\ell, k)}{MPK(\ell, k)} \overset{def}{=} TRS(\ell, k).$$

So what we call $TRS(\ell, k)$ is something that preserve the fact that $\Delta y = 0$ after dropping ${}^{\Delta k}\!/_{\Delta \ell}$, but we might as well use that dropped piece, because it is identical to $TRS(\ell, k)$ by definition.

Since we are given that choice

$$TRS(\ell, k) = -3$$

let's us assume that it is the optimal. Then by using "the dropped piece"

$$\frac{\Delta k}{\Delta \ell} = -3$$

which by rearranging gives

$$\Delta k = -3\Delta\ell$$

or, in words, if 1 extra unit of labor hired 3 units of capital can be dropped.

But we also know that opportunity cost of hiring labor is 2. Which could be understood as "hiring a unit of labor says no to "hiring" 2 unit of capital". In math this idea take form of this ration

$$\frac{w}{r} = 2$$

and if we set

$$r = 1$$

then

$$w = 2$$

in words, labor is twice more expensive.

So at point

$$TRS(\ell, k) \neq -\frac{w}{r}$$
$$-3 \neq -2$$

we can hire 1 unit of labor and spend \$2, but we drop 3 unit of capital which saves us \$3, thus we save \$1=(\$3-\$2). Which contradicts the fact that the point is optimal.

Question 5 (Understanding TRS and least-cost input combination): checklist

- Suppose a producer's technology is described by a Cobb-Douglas production function.
- Further, suppose that

  the opportunity cost of labor equal to 2

  and

  the producer is producing a desired quantity x

  by using a bundle of inputs at which

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- We know that in this case the producer would fail to minimize his production costs.
- Explain how the producer can save money by altering the composition of his current bundle of inputs.
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References

Nechyba, Thomas (2016). *Microeconomics: an intuitive approach with calculus*. Nelson Education. URL: https://sergeyvalexeev.files.wordpress.com/2018/03/1nechyba_t_microeconomics_an_intuitive_approach_with_calculus.pdf.

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